IJERMT

International Journal Of Engineering Research & Management Technology ISSN: 2348-4039

Email: editor@ijermt.org March 2014 Volume 1, Issue 2 Website: ijermt.org

Relationship between fractional calculus operators and Aleph function with special reference to Laplace Transform

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ABSTRACT
In this paper, we investigate how fractional derivatives and integrals can be possibly used in establishing a formula exhibiting relationship between more generalized special function named Aleph function and Laplace transform, which allows the straight forward derivation of some useful results involving fractional operators and Aleph function in terms of Mellin-Barnes Contour integral . Also some special cases has been discussed.

2010 Mathematics Subject Classifications: 26A33, 33E20, 33C20, 33C45

Key words: Aleph function, Fractional Calculus operators and Special functions.

1. INTRODUCTION AND PRELIMINARIES

The Aleph-function is defined in terms of the Mellin-Barnes type integral in the following manner [1, 2]

$$\aleph(z) = \left[\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[t\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots..[\tau_j(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m},\ldots..[\tau_j(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]\right]$$

$$\aleph(z) = \left[\frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s)\, t^s\, ds\right]$$

$$\Omega^{m,n}_{p_i,q_i,\tau_i;r}(s) = \frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_js\right).\prod_{j=1}^{n}\Gamma\left(1-a_j-A_js\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-B_{ji}s\right).\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+A_{ji}s\right)}.$$

Where $z\neq 0,\ i=\sqrt{-1}$.

An account of the convergence conditions for the defining integral can be found in the paper by Saxena and Pogány [1 (also see [3]).

We present below the definitions of the following fractional calculus operators of arbitrary order, a detailed account of fractional calculus operators can be found in the monograph by Samko et al. [4] and in a survey paper by Srivastava and Saxena [5] and Haubold-Mathai-Saxena [6].

In this section we present a brief sketch of various operators of fractional integration and fractional differentiation of arbitrary order. Among the various operators studied, included are the Riemann-Liouville fractional operators, Weyl operators and Saigo's operators etc. There exist more than one versions of the fractional integral operators. The fractional integral defined as follows

Riemann-Liouville fractional operators:

$I_{a,x}^{-\mu} f(x) = \frac{1}{(\mu-1)!}\int_a^x (x-y)_{\mu-1} f(y)dy$; $(x>0)$, $\mu>0$

Weyl Fractional Integral Operator

The Weyl fractional integral of f(x) of order α, is defined as

$W_\infty^\alpha f(x) = \frac{1}{\Gamma\alpha}\int_x^\infty (t-x)^{\alpha-1} f(t)\,dt$, $-\infty < x < \infty$

where $\alpha \in C$, $Re(\alpha)>0$, is also denoted by $I_x^\alpha f(x)$.

Kober Fractional Integral Operator

$$E_{0,x}^{\alpha,\eta} f(x) = \frac{(x)^{-\eta-\alpha}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} t^\eta f(t)dt;\ Re(\alpha) > 0$$

Saigo Fractional Integral Operator

$I_{0,x}^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma\alpha}\int_0^x (x-t)^{\alpha-1} {}_2F_1\left[\alpha+\beta, -\eta\,; 1-\frac{t}{x}\right] f(t)\,dt$, $Re(\alpha)>0$

Main results

In these sections, we will derive the Laplace transform of more generalized function of special functions named as aleph function.

Proposition: Laplace transform of more generalized special function named as Aleph function.

$$L\left[\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[t\left|\begin{matrix}(a_j,A_j)_{1,n},\dots\dots[\tau_j(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m},\dots\dots[\tau_j(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]\right] = L\left[\frac{1}{2\pi i}\int_L \Omega_{p_i,q_i,\tau_i;r}^{m,n}(s)\,t^s\,ds\right]$$

$$= \int_0^\infty e^{-pt}\left[\frac{1}{2\pi i}\int_L \theta(s)\,t^s\,ds\right]$$

$$= \frac{1}{2\pi i}\int_L \theta(s)\,\frac{\Gamma(s+1)}{p^{s+1}}\,ds$$

$$L\left[\aleph_{p_i,q_i,\tau_i;r}^{m,n}(t)\right] = \widehat{N}(t;p)$$

$$I_t^v\left[p^{-\lambda}\widehat{N}(t;p)\right] = I_t^v\left[p^{-\lambda}\frac{1}{2\pi i}\int_L \theta(s)\frac{\Gamma(s+1)}{p^{s+1}}\frac{1}{2\pi i}\,ds\right]$$

$$= \frac{1}{2\pi i}\frac{1}{\Gamma(v)}\int_0^t\left[\int_L (t-p)^{v-1}p^{-s-\lambda-1}\,\theta(s)\,\Gamma(s+1)ds\right]dp$$

In this section, we establish a theorem with the help of above proposition involving Laplace transform of Aleph function and fractional Calculus operators.

Results based on Riemann-Liouville Fractional operator

Theorem 1: Let $\mu > 0, \beta > 0, \lambda > 0$ and $a \in R$. Let I_{x+}^μ be the Riemann-Liouville operator. Then

$$I_{0^+}^{\alpha}\left[p^{-\lambda}\widehat{N}(t;p)\right] == p^{-\lambda+\alpha-1}\aleph_{p_i+2,\ q_i+1,\tau_i\,;\,r}^{m+1,n+1}\left[\frac{1}{p}\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots..[\tau_j(a_{ji},A_{ji})]_{n+1,p_i}(0,1),(\alpha-\lambda,1),\\ (b_j,B_j)_{1,m},\ldots..[\tau_j(b_{ji},B_{ji})]_{m+1,q_i}(-\lambda,1),\end{matrix}\right.\right]$$

Since, we know that

$$I_{0^+}^{\alpha}x^{\lambda} = x^{\lambda+\alpha}\frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda-\beta)}$$

This gives,

$$=\frac{1}{2\pi i}\int_L \theta(s)\ \Gamma(s+1)\left(I_{0^+}^{\alpha}p^{-\lambda-s-1}\right)ds$$

$$=\frac{1}{2\pi i}\int_L \theta(s)\,p^{-\lambda-s-1+\alpha}\frac{\Gamma(-\lambda-s)}{\Gamma(-\lambda+\alpha-s)}\Gamma(s+1)ds$$

$$= p^{-\lambda+\alpha-1}\aleph_{p_i+2,\ q_i+1,\tau_i\,;\,r}^{m+1,n+1}\left[\frac{1}{p}\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots..[\tau_j(a_{ji},A_{ji})]_{n+1,p_i}(0,1),(\alpha-\lambda,1),\\ (b_j,B_j)_{1,m},\ldots..[\tau_j(b_{ji},B_{ji})]_{m+1,q_i}(-\lambda,1),\end{matrix}\right.\right]$$

Theorem 2: Let $\mu > 0, \beta > 0, \lambda > 0$ and $a \in R$. Let I_{x-}^{μ} be the Riemann-Liouville operator. Then

$$I_{0^-}^{\alpha}\left[p^{-\lambda}\widehat{N}(t;p)\right] = I_{0^-}^{\alpha}\left[p^{-\lambda}\frac{1}{2\pi i}\int_L \theta(s)\ \frac{\Gamma(s+1)}{p^{s+1}}ds\right]$$

$$=\frac{1}{2\pi i}\int_L \theta(s)\ \Gamma(s+1)\left(I_{0^-}^{\alpha}p^{-\lambda-s-1}\right)ds$$

$$=\frac{1}{2\pi i}\int_L \theta(s)\,p^{-\lambda-s-1+\alpha}\frac{\Gamma(1-\alpha+\lambda+s)}{\Gamma(1+\lambda+s)}\Gamma(s+1)ds$$

$$= p^{-\lambda+\alpha-1}\aleph_{p_i+2,\ q_i+1,\tau_i\,;\,r}^{m,n+1}\left[\frac{1}{p}\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots..[\tau_j(a_{ji},A_{ji})]_{n+1,p_i}(0,1),(\alpha-\lambda,1),\\ (b_j,B_j)_{1,m},\ldots..[\tau_j(b_{ji},B_{ji})]_{m+1,q_i}(-\lambda,1),\end{matrix}\right.\right]$$

Results based on weyl Fractional operator

Theorem 3 : Let $\mu > 0, \beta > 0, \lambda > 0$ and $a \in R$. Let W_x^{μ} be the weyl operator. Then

$$W_x^{\mu}\left[p^{-\lambda}\widehat{N}(t;p)\right] = p^{\mu-\lambda-1}\frac{1}{2\pi i}\int \phi(s)\frac{\Gamma(1-\mu+\lambda+s)}{\Gamma(1+\lambda+s)}p^{-s}ds$$

$$= p^{\mu-\lambda-1}\aleph_{p_i,q_i,\tau_i\,;r}^{m,n}\left[t\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots..[\tau_j(a_{ji},A_{ji})]_{n+1,p_i}\\ (b_j,B_j)_{1,m},\ldots..[\tau_j(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]$$

$$W_x^{\mu}\left[p^{-\lambda}\frac{1}{2\pi i}\int_L \phi(s)\frac{\Gamma(s+1)}{p^{s+1}}\,ds\right]=\frac{1}{2\pi i}\int_l \Phi(s)\frac{\Gamma(1-\mu+\lambda\quad+s)}{\Gamma(1+\lambda+s)}p^{-\lambda-1-s+\mu}ds$$

$$=p^{\mu-\lambda-1}\frac{1}{2\pi i}\int_l\qquad\int_l \Phi(s)\frac{\Gamma(1-\mu+\lambda\quad+s)}{\Gamma(1+\lambda+s)\,p^s}ds$$

$$W_x^{\mu}\left[p^{-\lambda}\widehat{N}(t;p)\right]=p^{\mu-\lambda-1}\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[t\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots..[\tau_j(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m},\ldots..[\tau_j(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]$$

Results based on Saigo's Fractional operator

Lemma (i): If $Re(\gamma)>0, Re(\rho)>max[0, Re(\alpha+\alpha'+\beta'-\gamma), Re(\alpha'-\beta')]$, then

$$I_{0^+}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1}=x^{\rho-\alpha-\alpha'+\gamma-1}\frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)\Gamma(\rho+\beta')}$$

(ii) $Re(\gamma)>0, Re(\rho)<1+min[Re(-\beta), Re(\alpha+\alpha'-\gamma), Re(\alpha+\beta'-\gamma)]$, then

Theorem 4: If $Re(\gamma)>0, Re(\rho)>max[0, Re(\alpha+\alpha'+\beta'-\gamma), Re(\alpha'-\beta')]$ and $I_{0^-}^{\alpha,\alpha',\beta,\beta',\gamma}$ be the Saigo; s fractional operator, then

$$I_{0^-}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1}=x^{\rho-\alpha-\alpha'+\gamma-1}\frac{\Gamma(1+\alpha+\alpha'-\gamma-\rho)\Gamma(1+\alpha+\beta'-\gamma-\rho)\Gamma(1-\beta-\rho)}{\Gamma(1-\rho)\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\rho)\Gamma(1+\alpha-\beta-\rho)}$$

$$I_{0^+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[p^{-\lambda}\widehat{N}(t;p)\right]=I_{0^+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[p^{-\lambda}\frac{1}{2\pi i}\int_L \theta(s)\frac{\Gamma(s+1)}{p^{s+1}}ds\right]$$

$$=\frac{1}{2\pi i}\int_L \theta(s)\,\Gamma(s+1)\left(I_{0^+}^{\alpha,\alpha',\beta,\beta',\gamma}p^{-\lambda-s-1}\right)ds$$

$$=\frac{1}{2\pi i}\int_L \theta(s)\,p^{-\lambda-s-\alpha-\alpha'+\gamma-1}\frac{\Gamma(-\lambda-s)\Gamma(-\lambda-s+\gamma-\alpha-\alpha'-\beta)\Gamma(-\lambda-s+\beta'-\alpha')}{\Gamma(-\lambda-s+\gamma-\alpha-\alpha')\Gamma(-\lambda-s+\gamma-\alpha'-\beta)\Gamma(-\lambda-s+\beta')}\Gamma(s+1)ds$$

$$=p^{-\lambda-\alpha-\alpha'+\gamma-1}\times$$

$$\aleph_{p_i+4,q_i+3,\tau_i;r}^{m+3,n+1}\left[\frac{1}{p}\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots..[\tau_j(a_{ji},A_{ji})]_{n+1,p_i}(0,1),(\gamma-\lambda-\alpha-\alpha',1),(-\alpha'-\beta'-\lambda,1),(\beta'-\lambda,1)\\(b_j,B_j)_{1,m},\ldots..[\tau_j(b_{ji},B_{ji})]_{m+1,q_i}(-\lambda,1),(\gamma-\alpha-\alpha'-\beta-\lambda,1),(\beta'-\alpha'-\lambda,1),\end{matrix}\right.\right]$$

$$I_{0^-}^{\alpha,\alpha',\beta,\beta',\gamma}\left[p^{-\lambda}\widehat{N}(t;p)\right]=I_{0^-}^{\alpha,\alpha',\beta,\beta',\gamma}\left[p^{-\lambda}\frac{1}{2\pi i}\int_L \theta(s)\frac{\Gamma(s+1)}{p^{s+1}}ds\right]$$

$$=\frac{1}{2\pi i}\int_L \theta(s)\,\Gamma(s+1)\left(I_{0^-}^{\alpha,\alpha',\beta,\beta',\gamma}p^{-\lambda-s-1}\right)ds$$

$$=\frac{1}{2\pi i}\int_L \theta(s)\,p^{-\lambda-s-\alpha-\alpha'+\gamma-1}\frac{\Gamma(1+\alpha+\alpha'-\gamma+\lambda+s)\Gamma(1+\alpha+\beta'-\gamma+\lambda+s)\Gamma(1-\beta+\lambda+s)}{\Gamma(1+\lambda+s)\Gamma(1+\alpha+\alpha'+\beta'-\gamma+\lambda+s)\Gamma(1+\alpha-\beta+\lambda+s)}\Gamma(s+1)ds$$

$$= p^{-\lambda-\alpha-\alpha'+\gamma-1} \times$$

$$\aleph^{m,n+4}_{p_i+4,q_i+3,\tau_i;r}\left[\frac{1}{p}\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots..[\tau_j(a_{ji},A_{ji})]_{n+1,p_i}(0,1),(\alpha+\alpha'-\gamma+\lambda,1),(\alpha+\beta'-\gamma+\lambda,1),(\lambda-\beta,1)\\(b_j,B_j)_{1,m},\ldots..[\tau_j(b_{ji},B_{ji})]_{m+1,q_i}(\lambda,1),(\alpha+\alpha'+\beta'-\gamma+\lambda,1),(1+\alpha-\beta+\lambda,1),\end{matrix}\right.\right]$$

Special cases

By setting τ = 1 in above theorems, we get well-known results of I-function as reported in the papers of Jain et.al [8]

$$I^{\alpha}_{0^+}[p^{-\lambda}I(t;p)] = p^{-\lambda+\alpha-1}I^{m+1,n+1}_{p_i+2,\ q_i+1,r}\left[\frac{1}{p}\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots..[\tau_j(a_{ji},A_{ji})]_{n+1,p_i}(0,1),(\alpha-\lambda,1),\\(b_j,B_j)_{1,m},\ldots..[\tau_j(b_{ji},B_{ji})]_{m+1,q_i}(-\lambda,1),\end{matrix}\right.\right]$$

2. $W^{\mu}_{x}[p^{-\lambda}\hat{I}(t;p)] = p^{\mu-\lambda-1}\frac{1}{2\pi i}\int\phi(s)\frac{\Gamma(1-\mu+\lambda+s)}{\Gamma(1+\lambda+s)}\,p^{-s}ds$

$$= p^{\mu-\lambda-1}I^{m,n}_{p_i,q_i,\tau_i;r}\left[t\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots..[\tau_j(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m},\ldots..[\tau_j(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]$$

3. $I^{\alpha,\alpha',\beta,\beta',\gamma}_{0^+}[p^{-\lambda}\widehat{N}(t;p)]$

$$\aleph^{m,n+4}_{p_i+4,q_i+3,\tau_i;r}\left[\frac{1}{p}\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots..[\tau_j(a_{ji},A_{ji})]_{n+1,p_i}(0,1),(\alpha+\alpha'-\gamma+\lambda,1),(\alpha+\beta'-\gamma+\lambda,1),(\lambda-\beta,1)\\(b_j,B_j)_{1,m},\ldots..[\tau_j(b_{ji},B_{ji})]_{m+1,q_i}(\lambda,1),(\alpha+\alpha'+\beta'-\gamma+\lambda,1),(1+\alpha-\beta+\lambda,1),\end{matrix}\right.\right]$$

Conclusion

The results proved in this paper gives the evaluation of the Laplace transforms of the Aleph function functions in relation with the fractional operators. It has many applications in sciences and engineering for its special fundamental properties. In this connection one can refer to the work of Saxena, etal [1]

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